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# Stochastic comparisons of order statistics from gamma distributions

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## Abstract

Let  $(X_1, X_2, \dots, X_n)$  and  $(Y_1, Y_2, \dots, Y_n)$  be gamma random vectors with common shape parameter  $\alpha$  ( $0 < \alpha \leq 1$ ) and scale parameters  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $(\mu_1, \mu_2, \dots, \mu_n)$ , respectively. Let  $\mathbf{X}_{(0)} = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$ ,  $\mathbf{Y}_{(0)} = (Y_{(1)}, Y_{(2)}, \dots, Y_{(n)})$  be the order statistics of  $(X_1, X_2, \dots, X_n)$  and  $(Y_1, Y_2, \dots, Y_n)$ . Then  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  majorizes  $(\mu_1, \mu_2, \dots, \mu_n)$  implies that  $\mathbf{X}_{(0)}$  is stochastically larger than  $\mathbf{Y}_{(0)}$ . However if the common shape parameter  $\alpha > 1$ , we can only compare the *the first- and last-order statistics*. Some earlier results on stochastically comparing proportional hazard functions are shown to be special cases of our results.

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## 1. Introduction

In reliability theory and life testing, gamma distributions play an important role, and stochastic comparisons of these distributions have also found numerous applications (in the field of reliability, economic, comparison of experiments and so on). A lot of work exists in the literature on the stochastic properties of various statistics based on random samples. However, not much attention has been given to

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the case that the underlying random variables are independent but not identically distributed. Some interesting results on order statistics and spacings from heterogeneous populations have been obtained by Sen [19], Pledger and Proschan [17], Proschan and Sethuraman [18], Bapat and Beg [1], Bapat and Kochar [2], Kochar and Kirmani [8], Kochar and Korwar [9], Kochar and Rojo [11]; on convolution from independent but nonidentical life distributions by Boland et al. [4], Kochar and Ma [10], Khaledi and Kochar [6,7], Chang [5], Korwar [12]; on probability inequalities of nonlinear combinations from normal distributions by Bakirov [3]. A more common result can be found in [14].

Let  $X$  have a gamma distribution with shape parameter  $\alpha(>0)$  and scale parameter  $\lambda$ . Denote the density of  $X$  by  $f(x; \alpha, \lambda)$  and let  $F(x)$ ,  $\bar{F}(x) = 1 - F(x)$  and  $r(x) = f(x)/\bar{F}(x)$  be its distribution function, survival function and hazard rate function, respectively. Pledger and Proschan [17] and Proschan and Sethuraman [18] considered the problem of stochastically comparing the order statistics coming from independent and nonidentical exponential distributions. Kochar and Korwar [9], Kochar and Rojo [11] pursued this topic further and obtained some new results. In this paper, we extend the results of Proschan and Sethuraman [18] from exponential distributions to gamma distributions with common shape parameter  $\alpha \leq 1$ . Here are our main results:

**Theorem 1.1.** *Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  be independent gamma random vectors with common shape parameter  $\alpha$  ( $0 < \alpha \leq 1$ ) and scale parameters  $(\lambda_1, \dots, \lambda_n)$ ,  $(\mu_1, \dots, \mu_n)$ , respectively. Let  $X_{(i)}$ ,  $i = 1, 2, \dots, n$  be the  $i$ th-order statistics of  $\mathbf{X}$ , and  $Y_{(i)}$ ,  $i = 1, 2, \dots, n$  be the  $i$ th-order statistics of  $\mathbf{Y}$ . Set  $\mathbf{X}_{(0)} = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$  and  $\mathbf{Y}_{(0)} = (Y_{(1)}, Y_{(2)}, \dots, Y_{(n)})$ . If  $\lambda \succ \mu$ , then*

$$\mathbf{X}_{(0)} \geq_{st} \mathbf{Y}_{(0)}. \quad (1.1)$$

**Theorem 1.2.** *For random vectors  $\mathbf{X}$ ,  $\mathbf{Y}$  in Theorem 1.1, if the common shape parameter  $\alpha > 1$  and scale parameters satisfying  $\lambda \succ \mu$ , we have*

$$X_{(1)} \leq_{st} Y_{(1)}; \quad X_{(n)} \geq_{st} Y_{(n)}. \quad (1.2)$$

## 2. Preliminaries and the proof of main results

In this section, we first introduce some definitions and properties of the usual stochastic ordering, majorization and Schur-convex functions. Then we will prove our main results.

**Definition 2.1** (Shaked and Shanthikumar [20]). (i) Let  $X$  and  $Y$  be random variables,  $X$  is said to be larger than  $Y$  in the usual stochastic order if  $P(X \leq t) \leq P(Y \leq t)$  for all  $t \in \mathbf{R}$ ; in symbols,  $X \geq_{st} Y$ .

(ii) Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  be random vectors,  $\mathbf{X}$  is said to be larger than  $\mathbf{Y}$  in the usual stochastic order if  $P(\mathbf{X} \in U) \geq P(\mathbf{Y} \in U)$  for all upper set,<sup>1</sup> denoted by  $\mathbf{X} \geq_{st} \mathbf{Y}$ .

(iii) Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be a random vector. Suppose that for  $i = 2, 3, \dots, n$ ,

$$[X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1}] \leq_{st} [X_i | X_1 = x'_1, \dots, X_{i-1} = x'_{i-1}]$$

whenever  $x_j \leq x'_j$ ,  $j = 1, 2, \dots, i-1$ .

Then  $\mathbf{X}$  is said to be conditionally increasing in sequence (CIS).

It is usually difficult to check for usual multivariate stochastic ordering directly from its definition. However, if either  $\mathbf{X}$  or  $\mathbf{Y}$  is CIS, Theorem 4.B.4 [20, p. 117] gives sufficient conditions for the usual multivariate stochastic order by means of the usual univariate stochastic order. See [20] or [15] for detailed discussions on this and other equivalent conditions.

The next concept we need is that of majorization. It considers the diversity of the components of a vector. Majorization is useful and very powerful in deriving certain type of inequalities. For more details on majorization and its applications, see [13] or [16].

**Definition 2.2.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,  $\mu = (\mu_1, \dots, \mu_n)$  denote two real vectors. Let  $\lambda_{[1]} \geq \lambda_{[2]} \geq \dots \geq \lambda_{[n]}$ ,  $\mu_{[1]} \geq \mu_{[2]} \geq \dots \geq \mu_{[n]}$  be their ordered components. Then  $\mu$  is said to be majorized by  $\lambda$ , in symbols  $\lambda \succ \mu$ , if

$$\sum_{i=1}^m \lambda_{[i]} \geq \sum_{i=1}^m \mu_{[i]}$$

for  $m = 1, \dots, n-1$ , and  $\sum_{i=1}^n \lambda_{[i]} = \sum_{i=1}^n \mu_{[i]}$ .

If vector  $\lambda$  majorizes  $\mu$ , then there exists a finite number, say  $r$ , of vectors  $\lambda^{(1)}, \dots, \lambda^{(r)}$ , such that  $\lambda = \lambda^{(1)} \succ \dots \succ \lambda^{(r)} = \mu$  and such that  $\lambda^{(i)}$  and  $\lambda^{(i+1)}$  differ in two coordinates only,  $i = 1, 2, \dots, r-1$  (see [16], p. 321).

To prove the main result we need the following definition of Schur-concave functions and related results. For detail can also be found in [13] or [16].

**Definition 2.3.** A function  $\phi(\lambda) : \mathbf{R}^n \rightarrow \mathbf{R}$  is said to be a Schur-concave function if  $\phi(\lambda) \leq \phi(\mu)$  for all  $\lambda \succ \mu$ .

The following theorem provides sufficient and necessary conditions for the Schur-concavity. It is very useful in proving Schur-concavity.

<sup>1</sup>A set  $U \subseteq \mathbf{R}^n$  is called upper set if  $\mathbf{Y} \in U$  whenever  $\mathbf{Y} \geq \mathbf{X}$  and  $\mathbf{X} \in U$ .

**Theorem 2.4** (Marshall and Olkin [13]). *A permutation-symmetric differentiable function  $\phi(\mathbf{X})$  is Schur concave if and only if*

$$(X_i - X_j) \left( \frac{\partial \phi(\mathbf{X})}{\partial X_i} - \frac{\partial \phi(\mathbf{X})}{\partial X_j} \right) \leq 0$$

for all  $i \neq j$ .

The inequality is reversed for Schur convex functions.

Next we will prove our main results. We first prove Theorem 1.1 for the special case of bivariate gamma random vectors.

**Lemma 2.5.** *Let  $(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2)$ . Let  $X_1, X_2 (Y_1, Y_2)$  be independent gamma random variables with common shape parameter  $\alpha$  ( $0 < \alpha \leq 1$ ) and scale parameters  $\lambda_1, \lambda_2 (\mu_1, \mu_2)$ , respectively. Let  $X_{(1)} \leq X_{(2)} (Y_{(1)} \leq Y_{(2)})$  be the order statistics of  $X_1, X_2 (Y_1, Y_2)$ . Then*

$$(X_{(1)}, X_{(2)}) \geq_{st} (Y_{(1)}, Y_{(2)}).$$

**Proof.** Following Theorem 4.B.4 of [20, p. 117], it is sufficient to verify the three conditions: (i)  $X_{(1)} \geq_{st} Y_{(1)}$ , (ii)  $(X_1, X_2)$  is CIS, and (iii)  $(X_{(2)} | X_{(1)} = t_1) \geq_{st} (Y_{(2)} | Y_{(1)} = t_1)$ .

We first prove that  $X_{(1)} \geq_{st} Y_{(1)}$  by proving that  $\bar{F}_{\min}(t) = P(X_{(1)} \geq t) = P(X_1 \geq t, X_2 \geq t)$  is Schur convex with respect to  $\lambda_1, \lambda_2$ . For all  $t > 0$

$$\frac{\partial \bar{F}_{\min}}{\partial \lambda_1} = \frac{1}{\Gamma(\alpha)} P(X_2 \geq t) (-\lambda_1^{\alpha-1} t^\alpha e^{-\lambda_1 t}), \quad (2.1)$$

$$\frac{\partial \bar{F}_{\min}}{\partial \lambda_2} = \frac{1}{\Gamma(\alpha)} P(X_1 \geq t) (-\lambda_2^{\alpha-1} t^\alpha e^{-\lambda_2 t}). \quad (2.2)$$

It is well-known that the hazard rate function of the gamma distribution,  $r(t)$ , is decreasing in  $t$  when  $0 < \alpha \leq 1$ . So we have

$$r(\lambda) = \frac{\lambda^{\alpha-1} t^\alpha e^{-\lambda t}}{\int_{\lambda}^{+\infty} x^{\alpha-1} t^\alpha e^{-tx} dx}$$

is decreasing in  $\lambda$ . From (2.1) and (2.2) we can show that

$$(\lambda_1 - \lambda_2) \left( \frac{\partial \bar{F}_{\min}}{\partial \lambda_1} - \frac{\partial \bar{F}_{\min}}{\partial \lambda_2} \right) \geq 0.$$

According to Theorem 2.4, we have

$$X_{(1)} \geq_{st} Y_{(1)}. \quad (2.3)$$

Next we prove  $(X_{(1)}, X_{(2)})$  is CIS. Since

$$\begin{aligned} & \frac{\int_{t_2-t_1}^{+\infty} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx + \int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx}{\int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx} \\ &= 1 + \frac{\int_{t_2-t_1}^{+\infty} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx}{\int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx} \\ &= 1 + \frac{\int_{t_2}^{+\infty} x^{\alpha-1} [e^{-\lambda_1(x-t_1)} + e^{-\lambda_2(x-t_1)}] dx}{\int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx} \end{aligned}$$

is increasing in  $t_1$  for all  $0 < t_1 \leq t_2$  and  $0 < \alpha \leq 1$ . We have the conditional distribution

$$\begin{aligned} & P(X_{(2)} \leq t_2 \mid X_{(1)} = t_1) \\ &= \frac{\int_{t_1}^{t_2} x^{\alpha-1} [e^{-\lambda_1(x-t_1)} + e^{-\lambda_2(x-t_1)}] dx}{\int_{t_1}^{+\infty} x^{\alpha-1} [e^{-\lambda_1(x-t_1)} + e^{-\lambda_2(x-t_1)}] dx} \\ &= \frac{\int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx}{\int_{t_2-t_1}^{+\infty} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx + \int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx} \end{aligned}$$

decreasing in  $t_1$ . That  $(X_{(1)}, X_{(2)})$  is CIS, so is  $(Y_{(1)}, Y_{(2)})$ .

Finally, we prove that

$$[X_{(2)} \mid X_{(1)} = t_1] \geq_{st} [Y_{(2)} \mid Y_{(1)} = t_1]. \quad (2.4)$$

Let  $F(t_2 \mid t_1) = P(X_{(2)} \leq t_2 \mid X_{(1)} = t_1)$ , It is sufficient to prove that  $F(t_2 \mid t_1)$  is Schur concave with respect to  $\lambda_1, \lambda_2$ . Let  $\varphi(t) = \int_0^t (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx$ , we have

$$\begin{aligned} & \frac{F(t_2 \mid t_1)}{\partial \lambda_1} = \\ & \frac{\varphi(t_2 - t_1) \int_0^{+\infty} x(x+t_1)^{\alpha-1} e^{-\lambda_1 x} dx - \varphi(+\infty) \int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} e^{-\lambda_1 x} dx}{\varphi^2(+\infty)}, \end{aligned}$$

$$\begin{aligned} & \frac{F(t_2 \mid t_1)}{\partial \lambda_2} = \\ & \frac{\varphi(t_2 - t_1) \int_0^{+\infty} x(x+t_1)^{\alpha-1} e^{-\lambda_2 x} dx - \varphi(+\infty) \int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} e^{-\lambda_2 x} dx}{\varphi^2(+\infty)}. \end{aligned}$$

Then

$$\begin{aligned}
 & \frac{\partial F(t_2 | t_1)}{\partial \lambda_1} - \frac{\partial F(t_2 | t_1)}{\partial \lambda_2} \\
 &= \frac{\varphi(+\infty) \int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx}{\varphi^2(+\infty)} \\
 & \quad - \frac{\varphi(t_2-t_1) \int_0^{+\infty} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx}{\varphi^2(+\infty)} \\
 &= \frac{\int_{t_2-t_1}^{+\infty} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx}{\varphi^2(+\infty)} \frac{\int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx}{\varphi^2(+\infty)} \\
 & \quad - \frac{\int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx}{\varphi^2(+\infty)} \frac{\int_{t_2-t_1}^{+\infty} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx}{\varphi^2(+\infty)}. \tag{2.5}
 \end{aligned}$$

Since  $\varphi^2(+\infty) > 0$ , it is sufficient to compare the numerator. Let  $\lambda_1 > \lambda_2$ , the proof for the case when  $\lambda_1 < \lambda_2$  follows from the same kind of arguments. The first term in (2.5) can be written as

$$\begin{aligned}
 & \int_{t_2-t_1}^{+\infty} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx \int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx \\
 &= \lim_{A \rightarrow +\infty} \int_{t_2-t_1}^A (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx \\
 & \quad \times \int_0^{t_2-t_1} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx \\
 &= \lim_{A \rightarrow +\infty} (\xi_1 + t_1)^{\alpha-1} \left[ \left( \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} + \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} \right) - \left( \frac{1}{\lambda_1} e^{-\lambda_1 A} + \frac{1}{\lambda_2} e^{-\lambda_2 A} \right) \right] \\
 & \quad \times \eta_1(\eta_1 + t_1)^{\alpha-1} \left[ \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) - \left( \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} - \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} \right) \right] \tag{2.6}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{A \rightarrow +\infty} (\xi_1 + t_1)^{\alpha-1} \eta_1(\eta_1 + t_1)^{\alpha-1} \left[ \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \left( \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} + \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} \right) \right. \\
 & \quad \left. - \left( \frac{1}{\lambda_2^2} e^{-2\lambda_2(t_2-t_1)} - \frac{1}{\lambda_1^2} e^{-2\lambda_1(t_2-t_1)} \right) \right], \tag{2.7}
 \end{aligned}$$

where  $\xi_1 \in (t_2 - t_1, A)$  and  $\eta_1 \in (0, t_2 - t_1)$ . The second term in (2.5) can be written as

$$\begin{aligned} & \int_0^{t_2-t_1} (x+t_1)^{\alpha-1} [e^{-\lambda_1 x} + e^{-\lambda_2 x}] dx \int_{t_2-t_1}^{+\infty} x(x+t_1)^{\alpha-1} [e^{-\lambda_2 x} - e^{-\lambda_1 x}] dx \\ &= \lim_{B \rightarrow +\infty} (\xi_2 + t_1)^{\alpha-1} \eta_2 (\eta_2 + t_1)^{\alpha-1} \left[ \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_1} \right) \left( \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} - \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} \right) \right. \\ & \quad \left. - \left( \frac{1}{\lambda_2^2} e^{-2\lambda_2(t_2-t_1)} - \frac{1}{\lambda_1^2} e^{-2\lambda_1(t_2-t_1)} \right) \right], \end{aligned} \quad (2.8)$$

where  $\xi_2 \in (0, t_2 - t_1)$  and  $\eta_2 \in (t_2 - t_1, B)$ . Using these observations, it follows that

$$(\xi_1 + t_1)^{\alpha-1} \leq (\xi_2 + t_1)^{\alpha-1} \quad \text{and} \quad \eta_1 (\eta_1 + t_1)^{\alpha-1} \leq \eta_2 (\eta_2 + t_1)^{\alpha-1}.$$

Since  $\lambda_1 > \lambda_2$ , we get

$$\begin{aligned} & \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \left( \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} + \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} \right) \\ & \leq \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_1} \right) \left( \frac{1}{\lambda_2} e^{-\lambda_2(t_2-t_1)} - \frac{1}{\lambda_1} e^{-\lambda_1(t_2-t_1)} \right). \end{aligned}$$

From (2.5), (2.7) and (2.8) we can show that

$$(\lambda_1 - \lambda_2) \left( \frac{\partial F(t_2 | t_1)}{\partial \lambda_1} - \frac{\partial F(t_2 | t_1)}{\partial \lambda_2} \right) \leq 0.$$

That is  $F(t_2 | t_1)$  is Schur concave with respect to  $\lambda_1, \lambda_2$ , so

$$[X_{(2)} | X_{(1)} = t_1] \geq_{st} [Y_{(2)} | Y_{(1)} = t_1].$$

Now the required result follows from Theorem 4.B.4 of [20].

$$(X_{(1)}, X_{(2)}) \geq_{st} (Y_{(1)}, Y_{(2)}). \quad \square$$

In order to prove Theorem 1.1, we need the following well-known result.

**Theorem 2.6** (Proschan and Sethuraman [18]). *Let  $\mathbf{a}$  and  $\mathbf{b}$  be two vectors of constants such that  $a_i \geq b_i, i = 1, \dots, n$ . Then  $a_{(i)} \geq b_{(i)}$ , for all  $i = 1, \dots, n$ .*

Now we prove the main result as stated in Theorem 1.1. This proof is analogous to that of Theorem 3.4 of Proschan and Sethuraman [19].

**Proof of Theorem 1.1.** Since  $\lambda \succ \mu$ , there exist  $r$  vectors  $\lambda^{(1)}, \dots, \lambda^{(r)}$  such that  $\lambda = \lambda^{(1)} \succ \lambda^{(2)} \succ \dots \succ \lambda^{(r)} = \mu$ , and  $\lambda^{(i)}, \lambda^{(i+1)}$  differ only in two components,  $i = 1, \dots, r-1$ . Thus in order to prove Theorem 1.1, we may, without loss of generality, assume

that  $\lambda, \mu$  differ only in two components, and in fact, assume  $\lambda_3 = \mu_3, \dots, \lambda_n = \mu_n$  and  $(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2)$ . Let

$$U_{\max} = \text{Max}(X_1, X_2), \quad U_{\min} = \text{Min}(X_1, X_2),$$

$$V_{\max} = \text{Max}(Y_1, Y_2), \quad V_{\min} = \text{Min}(Y_1, Y_2)$$

and  $W_3, W_4, \dots, W_n$  are independently distributed according to gamma distributions with common shape parameter  $\alpha$  and scale parameters  $\lambda_3, \dots, \lambda_n$ , respectively, and independently of  $(U_{\max}, U_{\min}, V_{\max}, V_{\min})$ . Then  $\mathbf{X}' = (U_{\max}, U_{\min}, X_3, \dots, X_n) =_{st} (U_{\max}, U_{\min}, W_3, \dots, W_n)$  and  $\mathbf{Y}' = (V_{\max}, V_{\min}, Y_3, \dots, Y_n) =_{st} (V_{\max}, V_{\min}, W_3, \dots, W_n)$ , and  $(U_{\max}, U_{\min}, W_3, \dots, W_n) \geq_{st} (V_{\max}, V_{\min}, W_3, \dots, W_n)$ . Theorem 4.B.1 of [20, p. 115] implies that there exist two random vectors  $\hat{\mathbf{X}}$  and  $\hat{\mathbf{Y}}$ , defined on the same probability space  $\Omega$ , such that

$$\hat{\mathbf{X}} =_{st} (U_{\max}, U_{\min}, W_3, \dots, W_n) =_{st} \mathbf{X}',$$

$$\hat{\mathbf{Y}} =_{st} (V_{\max}, V_{\min}, W_3, \dots, W_n) =_{st} \mathbf{Y}'$$

and

$$P(\hat{\mathbf{X}} \geq \hat{\mathbf{Y}}) = 1.$$

Theorem 2.6 implies that  $P(\hat{X}_{(1)} \geq \hat{Y}_{(1)}, \dots, \hat{X}_{(n)} \geq \hat{Y}_{(n)}) = 1$ , that is,

$$\mathbf{X}'_0 =_{st} \hat{\mathbf{X}}_0 \geq_{st} \hat{\mathbf{Y}}_0 =_{st} \mathbf{Y}'_0.$$

Note that  $\mathbf{X}'_0 =_{st} \mathbf{X}_0$  and  $\mathbf{Y}'_0 =_{st} \mathbf{Y}_0$ . Thus  $\mathbf{X}_0 \geq_{st} \mathbf{Y}_0$ .  $\square$

Based on the same derivation as in Lemma 2.5, the proof of Theorem 1.2 follows the fact that the hazard function  $r(t)$  of gamma distribution is increasing in  $t$  when the shape parameter  $\alpha > 1$ .

**Proof of Theorem 1.2.** Assume  $\lambda_3 = \mu_3, \dots, \lambda_n = \mu_n$  and  $(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2)$ .

We first prove  $X_{(1)} \leq_{st} Y_{(1)}$ , it is sufficient to prove  $\bar{F}_{\min}(t) = P(X_{(1)} \geq t) = P(X_1 \geq t, X_2 \geq t, \dots, X_n \geq t)$  is Schur concave with respect to  $\lambda_1, \lambda_2$  for all  $t > 0$

$$\frac{\partial \bar{F}_{\min}}{\partial \lambda_1} = \frac{1}{\Gamma(\alpha)} P(X_3 \geq t, \dots, X_n \geq t) P(X_2 \geq t) (-\lambda_1^{\alpha-1} t^\alpha e^{-\lambda_1 t}),$$

$$\frac{\partial \bar{F}_{\min}}{\partial \lambda_2} = \frac{1}{\Gamma(\alpha)} P(X_3 \geq t, \dots, X_n \geq t) P(X_1 \geq t) (-\lambda_2^{\alpha-1} t^\alpha e^{-\lambda_2 t}).$$

Since  $r(t)$ , the hazard rate function of gamma distribution, is increasing in  $t$  when shape parameter  $\alpha > 1$ . So we have

$$r(\lambda) = \frac{\lambda^{\alpha-1} t^\alpha e^{-\lambda t}}{\int_{\lambda}^{+\infty} x^{\alpha-1} t^\alpha e^{-tx} dx}$$



is increasing in  $\lambda$ . It is easy to show that

$$(\lambda_1 - \lambda_2) \left( \frac{\partial \bar{F}_{\min}}{\partial \lambda_1} - \frac{\partial \bar{F}_{\min}}{\partial \lambda_2} \right) \leq 0.$$

By Theorem 2.4, we have

$$X_{(1)} \leq_{st} Y_{(1)}.$$

Similarly, we can show that  $F_{\max}(t) = P(X_1 \leq t, X_2 \leq t, \dots, X_n \leq t)$  is Schur concave with respect to  $\lambda_1, \lambda_2$  for all  $t \geq 0$ . So

$$X_{(n)} \geq_{st} Y_{(n)}. \quad \square$$

### 3. Corollaries and remarks

We get the following interesting corollary.

**Corollary 3.1.** *Under the conditions of Theorem 1.1,*

$$\sum_{i=1}^n c_i X_{(i)} \geq_{st} \sum_{i=1}^n c_i Y_{(i)},$$

for all  $c_i \geq 0, i = 1, \dots, n$ . Thus

$$\sum_{i=1}^k X_{(i)} \geq_{st} \sum_{i=1}^k Y_{(i)}, \quad k = 1, \dots, n,$$

in particular,

$$\sum_{i=1}^n X_{(i)} \geq_{st} \sum_{i=1}^n Y_{(i)},$$

extending the result of Korwar [12] to the case when  $\alpha \leq 1$ .

**Proof.** Based on the fact that  $\phi := \sum_{i=1}^n c_i x_i$  is a increasing function if  $c_i \geq 0, i = 1, 2, \dots, n$ . This follows immediately from the equivalent conditions of the Definition 2.1 [20, p. 4].  $\square$

**Remark 3.2.** Chi-square distribution, one degree of freedom is a gamma random variable with  $\alpha = \frac{1}{2}$ . Bakirov [3] have proved  $Pr(\sum_{i=1}^n \lambda_i \xi_i^2 \leq x) \leq Pr(\sum_{i=1}^n \mu_i \xi_i^2 \leq x)$  for all  $x \geq 2$ , where  $\xi_i, i = 1, 2, \dots, n$  is standard normal distribution and  $\lambda \succ \mu$ . However, according to Corollary 3.1 we can obtain  $Pr(\sum_{i=1}^n \frac{1}{\lambda_i} \xi_i^2 \leq x) \leq Pr(\sum_{i=1}^n \frac{1}{\mu_i} \xi_i^2 \leq x)$  for all  $x$ .

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